Math 564: Advance Analysis 1
Lecture 9

Proof of (6) (continued). We do the same trick as before: shrink $(a,b]$ to make it compact and extend $(a_n,b_n]$ to make it open. More precisely, fix $\varepsilon>0$, and let $a<a'<a+\delta$ where δ is small enough so that $f(a')\approx_\varepsilon f(a)$. Similarly, let $b_n<b_n'<b_n+\delta_n$, where δ_n is small enough so that $f(b_n')\approx_{\varepsilon/2^{n+1}} f(b_n)$. There are such δ and (δ_n) by the right-continuity of f. Then $[a',b]\subseteq\bigcup_n (a_n,b_n')$ and by the compactness of $[a',b]$, there is N s.t. $[a',b]\subseteq\bigcup_{n<N}(a_n,b_n')$. Thus,

$$\mu_f((a,b])\approx_\varepsilon \mu_f((a',b])\overset{\text{finite subadd}}{\leq}\sum_{n<N}\mu_f((a_n,b_n'])\leq\sum_{n\in\mathbb{N}}\mu_f((a_n,b_n'])\approx_\varepsilon\sum_{n\in\mathbb{N}}\mu_f((a_n,b_n]).$$

$\square$

Thus, we have classified/fully described all Borel measures on $\mathbb{R}$ which are finite on bdd sets.

Measurable functions.
Let $(X,\mathcal{I})$ and $(Y,\mathcal{J})$ be measurable spaces.

Def.
- A function $f:X\to Y$ is called $(\mathcal{I},\mathcal{J})$-measurable if $f^{-1}(\mathcal{J})\subseteq\mathcal{I}$, i.e. for each $J\in\mathcal{J}$ $f^{-1}(J)\in\mathcal{I}$.
- When X and Y are top. spaces (without any given $\mathcal{I}$ and $\mathcal{J}$), we call a function $f:X\to Y$ Borel if it is $(\mathcal{B}(X),\mathcal{B}(Y))$-measurable, i.e. the f-preimage of each Borel set is Borel.
- When $(X,\mathcal{S},\mu)$ is a measure space and Y is a top. space, we call a function $f:X\to Y$ μ-measurable if f is $(\mathrm{Meas}_\mu,\mathcal{B}(Y))$-measurable, i.e. the f-preimage of each Borel set is μ-measurable.

Remark. The definition of μ-measurable is asymmetric, having a "smaller" σ-algebra on the right in order to get a larger class of functions which still are well-behaved. In particular, we want continuous functions to be measurable.

Prop. Continuous functions between top. spaces are Borel.

Proof. Let X, Y be top. spaces and $f: X \to Y$ be continuous, i.e. f-preimages of open sets are open. We do a top-down proof: Let $\mathcal{S} := \{B \in \mathcal{B}(Y) : f^{-1}(B) \text{ is Borel}\}$. Then $\mathcal{S}$ contains all open sets and is closed under complements and ctbl unions because f^{-1} commutes with these operations (unlike f), so $\mathcal{S} = \mathcal{B}(Y)$. $\square$

Caution: Compositions of μ-measurable functions are typically not μ-measurable. More precisely, let (X, μ), (Y, ν), (Z, ρ) be top. spaces equipped with Borel measures, e.g. all $= (\mathbb{R}, \lambda)$. Then for a μ-meas. $f: X \to Y$ and a ν-meas. function $g: Y \to Z$, there is no reason why $g \circ f$ should be μ-measurable because for a Borel $B \subseteq Z$, $g^{-1}(B)$ is ν-measurable so we have no control over $f^{-1}(g^{-1}(B))$.
A construction of an example where indeed $g \circ f$ is not μ-measurable will be outlined in HW.

This motivates the following definition:

Def. Let X, Y be top. spaces. A function $f: X \to Y$ is called *universally measurable* if it is μ-measurable for all Borel probability measures on X.

Prop. Composition of universally measurable functions is universally measurable.

Proof. HW.

The following proposition is, in a way, why measure theory started because ptwise limits Riemann integrable functions are usually not Riemann integrable, and ptwise limits of continuous functions are usually not continuous.

In separable metric spaces

Prop. Pointwise limits of μ-measurable functions are μ-measurable. More precisely, for a measure space (X, μ) and a sep. metric space Y, if the functions $f_n : X \to Y$ are μ-measurable and $f_n \xrightarrow[n\to\infty]{} f : X \to Y$ pointwise (i.e. $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$),

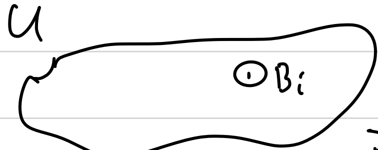
then f is μ-measurable.

Proof. Again top-down thinking: let $\mathcal{S} := \{B \in \mathcal{B}(Y) : f^{-1}(B) \in \mathrm{Meas}_\mu\}$. This is clearly a σ-alg. because Meas_μ is and f^{-1} commutes with complement and ctbl unions. It remains to show that $\mathcal{S}$ contains all open sets and hence $\mathcal{S} = \mathcal{B}(Y)$.

Let $U \subseteq Y$ be open. For each $x \in X$, if $f(x) \in U$, then $\forall^\infty n \in \mathbb{N}$ $f_n(x) \in U$. The converse is not true, but very close.

Let $U = \bigcup_{i \in \mathbb{N}} B_i$, where each B_i is an ball s.t. $\overline{B_i} \subseteq U$.

Then



$$f^{-1}(U) = \{x \in X : f(x) \in U\} = \bigcup_{i \in \mathbb{N}} \{x \in X : \forall^\infty_n f_n(x) \in \overline{B_i}\} = \bigcup_{i \in \mathbb{N}} \{x \in X : \forall^\infty_n x \in f_n^{-1}(\overline{B_i})\}$$ because:

$\Rightarrow$: If $f(x) \in U$ then $\exists i \in \mathbb{N}$ s.t. $f(x) \in B_i$, so $\forall^\infty_n f_n(x) \in B_i$.

$\Leftarrow$: If $\exists i \in \mathbb{N}$ s.t. $\forall^\infty_n f_n(x) \in \overline{B_i}$ then $f(x) = \lim_n f_n(x) \in \overline{B_i}$.

But for each fixed $i \in \mathbb{N}$ $\{x \in X : \forall_n^\infty \; x \in f_n^{-1}(\overline{B_i})\} = \liminf f_n^{-1}(\overline{B_i}) = \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} f_n^{-1}(\overline{B_i})'$, which is μ-meas hence $f_n^{-1}(\overline{B_i})$ is. $\square$

False

<u>Detour to regularity of Borel measures.</u> We prove a slightly stronger version of regularity for σ-finite measures.

<u>Regularity Revisited.</u> Let X be a metric space and μ be a σ-finite Borel measure on X. Then μ is strongly regular, i.e. for each μ-meas set $A \subseteq X$,

$$\mu(A) = \inf\{\mu(U) : A \subseteq U \text{ open}\}$$
$$= \sup\{\mu(C) : A \supseteq C \text{ closed}\}$$

Moreover, $\forall \varepsilon > 0$ $\exists$ open $U \supseteq A$ with μ .

Proof. We've already proven $(*)$ for finite measures w/ the last statement with ε is automatic. In general for a σ-finite measure μ, let $X = \bigcup_n X_n$ where X_n are Borel & $\mu(X_n) < \infty$.

(a) To prove the approximation by open sets, we replace each X_n with $U_n \supseteq X_n$ open with $\mu(U_n) < \infty$. Then let $U_n \supseteq V_n \supseteq A \cap U_n$ be an open set with $\mu(V_n \setminus (A \cap U_n)) \leq \varepsilon/2^{n+1}$. Then $\bigcup_n V_n \supseteq \bigcup_n A \cap U_n = A$ and $\mu(\bigcup_n V_n \setminus A) \leq \sum_n \varepsilon/2^{n+1} < \varepsilon$.

(b) We show as in (a) that WLOG, we could have taken X_n to be closed.

See Lecture 7, Corollary of regularity.